# A Correspondence Between Modules and Vector Bundles

Jacob Wood

## 1 Introduction

An important idea in mathematics is being able to study objects from multiple perspectives. The way mathematical objects interact with each other can reveal properties about those objects. Therefore, being able to describe things in multiple ways can be useful, as this gives us more tools with which to understand them. In algebraic geometry, there is a persistent connection between geometric spaces and algebraic structures. In this paper, we focus on one particular instance of this phenomenon, the correspondence between the vector bundles over the spectrum of a ring and modules over the ring. In particular, we will show how one can produce a vector bundle over the spectrum of a ring from a module over that ring.

## 2 Preliminaries

**Definition.** *Let $X$ be a topological space. A* **vector bundle** *is a triple $(E, X, \pi)$, where $E$ is a space and $\pi$ is a surjective function from $E$ to $X$ such that $\pi^{-1}(x)$ is a vector space for all $x \in X$. In other words, the fibre over each point is a vector space. We say that a vector bundle has rank $n$ if all of the fibres are $n-$dimensional vector spaces.*

Vector bundles are important for studying properties of the space $X$. In algebraic geometry, we study functions over a space, and vector bundles allow us to examine geometric properties of a space from a more algebraic perspective. The following notion describes a space of particular interest.

**Definition.** *The* **spectrum** *of a ring $R$, denoted $Spec(R)$ is the set of prime ideals of $R$.*

When $R = S[x_1, ..., x_n]$ for some ring $S$, we also denote this space $\mathbb{A}^n_S := Spec(S[x_1, ..., x_n])$. This is also called affine $n-$space over $S$. We say that elements $r \in R$ are functions on $Spec(R)$. The value of a function $r \in R$ at the point $\mathfrak{p} \in Spec(R)$ is $r$ mod $\mathfrak{p}$. The spectrum comes with a topology called the Zariski topology. The closed sets of the Zariski topology are the sets $V(I) = \{\mathfrak{p} \in Spec(R) | I \subset \mathfrak{p}\}$, the collection of prime ideals containing an ideal $I$ of $R$.

**Example.** $R = \mathbb{C}[x]$

As described above, $Spec(\mathbb{C}[x]) = \mathbb{A}^1_{\mathbb{C}}$ is the set of prime ideals of $\mathbb{C}[x]$. Notice that because $\mathbb{C}$ is an algebraically closed field, every polynomial of degree greater than zero can be factored into degree 1 polynomials, so every non-unital element is in an ideal generated by $(x - z)$ for some $z \in \mathbb{C}$. Since $\mathbb{C}[x]/(x - z) \cong \mathbb{C}$, these ideals are maximal, which implies that they are prime. The ideals of the form $(x - z)$ along with the zero ideal are the prime ideals of $\mathbb{C}[x]$. Therefore, the prime ideals of $\mathbb{C}[x]$ correspond to elements $z \in \mathbb{C}$.

**Example.** $R = \mathbb{C}[x, y]$

Now look at when $R = \mathbb{C}[x, y]$. Just like in the previous example, elements of $\mathbb{C}^2$ correspond to prime ideals in that the ideal $(x - z_1, y - z_2)$ is a prime ideal. However, there are other prime ideals. One example is $(y - x^2)$. In fact, if $f(x, y)$ is irreducible, then $(f(x, y))$ is prime. Therefore, $Spec(R)$ looks like $\mathbb{C}^2$ but with some additional points that correspond to irreducible polynomials in $\mathbb{C}[x, y]$. The prime ideals that have a corresponding point in $\mathbb{C}^2$ are the closed points of $Spec(\mathbb{C}[x, y])$. As it turns out, the ideals associated to the closed points are not just prime, they are are maximal. When we talk about vector bundles, we will reduce our discussion to the fibres over closed points.

**Example.** $R = k$, *where $k$ is a field.*

A field only has one prime ideal, the zero ideal, so $Spec(k) = \{(0)\}$. We will not prove it here, but there is a correspondence between vector bundles over $Spec(R)$ and $R-$modules. As a result of this correspondence, we are able to study vector bundles over $Spec(R)$ by way of studying the modules over $R$. In the following section, we will not show the correspondence both ways, but we will only go from a module to a vector bundle.

# 3 Constructing a Vector Bundle

Given a free, finitely generated $R-$module $M$ of rank $n$, we wish to construct the corresponding rank $n$ vector bundle $E$ over $Spec(R)$. In order to do this, we need for $M$ to be able to admit a basis for a vector space. We achieve this in the following way. Taking the dual of $M$ and examining the symmetric algebra over the dual, we get a ring that is a direct sum of $R-$modules. Explicitly,

$$Sym(M^*) = R \oplus M^* \oplus (M^*)^{\otimes 2} \oplus (M^*)^{\otimes 3} \oplus ...,$$

which is a graded ring over $R$. Notice that each of the factor rings in $Sym(M^*)$ is a tensor power of $M$. The equivalence relation $x \otimes y \sim y \otimes x$ in the symmetric algebra ensures that the basis elements behave like indeterminants in a polynomial. Furthermore, the grading on $Sym(M^*)$ means that each element only has finitely many non-zero terms. Therefore, if $M$ is a finitely generated free $R-$module of rank $n$, then $Sym(M^*) \cong R[x_1, ..., x_n]$. Now if we look at $Spec(Sym(M^*))$, we can see that this is just $Spec(R[x_1, ..., x_n]) = \mathbb{A}^n_R$, affine $n-$space over $R$! From this point, the idea is to find a function from $E = Spec(Sym(M^*))$ to $X = Spec(R)$. The way we do this is by finding a function from $R$ to $Sym(M^*)$. There is a natural function from $R$ to $Sym(M^*) \cong R[x_1, ..., x_n]$, namely the inclusion map $i$ such that $r \mapsto r \oplus 0 \oplus 0 \oplus ...$, which corresponds to a constant polynomial in $R[x_1, ..., x_n]$. To get a map from $E$ to $X$, we must look at the inverse images $i^{-1}(\mathfrak{q})$ of the prime ideals $\mathfrak{q} \in E$. Notice that if $ab \in i^{-1}(\mathfrak{q})$, then $i(ab) \in \mathfrak{q}$, so $i(a) \in \mathfrak{q}$ or $i(b) \in \mathfrak{q}$. Thus, $a \in i^{-1}(\mathfrak{q})$ or $b \in i^{-1}(\mathfrak{q})$, making $i^{-1}(\mathfrak{q})$ a prime ideal of $R$. In this way, $i^{-1}$ takes prime ideals of $Sym(M^*)$ to prime ideals in $R$. In other words, this is a function from $E = Spec(Sym(M^*))$ to $X = Spec(R)$. The fibre over a closed point $\mathfrak{p} \in X$ contains the prime ideals $\mathfrak{q} \in E$ which map to $\mathfrak{p}$. These consist of polynomials over $R/\mathfrak{p}$ in $n$ variables ($R/\mathfrak{p}[x_1, ..., x_n]$). Because $\mathfrak{p}$ is maximal, the ring $R/\mathfrak{p}$ is actually a field. Therefore, we have that the fibre over $\mathfrak{p}$ is the set of prime ideals of $R/\mathfrak{p}[x_1, ..., x_n]$, which we know to be $Spec(R/\mathfrak{p}[x_1, ..., x_n])$, an $n-$dimensional vector space over $R/\mathfrak{p}$. Consider the example of when $R = k$ and the module $M = k$ is viewed as a vector space over $k$. We have that $E = Spec(Sym(k^*)) \cong Spec(Sym(k)) = Spec(k[x]) = \mathbb{A}^1_k$. Since $X = Spec(R) = Spec(k) = \{(0)\}$, there is only one possible map from $E$ to $X$. The vector space ($k-$module) $k$ is 1-dimensional, and the corresponding vector bundle has rank 1.